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THE TOPOLOGICAL CENTERS OF MODULE ACTIONS

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Abstract. In this article, for Banach left and right module actions, we will extend some propositions from Lau and *Ülger* into general situations and we establish the relationships between topological centers of module actions. We also introduce the new concepts as Lw^*w-property and Rw^*w-property for Banach $A-bimodule$ B and we investigate the relations between them and topological center of module actions. We have some applications in dual groups.

1.Introduction and Preliminaries

As is well-known [1], the second dual A^{**} of A endowed with the either Arens multiplications is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in [6, 8, 13, 14, 15, 16, 17, 21, 22], and they have attracted by some attentions.

Now we introduce some notations and definitions that we used throughout this paper. Let A be a Banach algebra. We say that a net $(e_\alpha)_{\alpha\in I}$ in A is a left approximate identity $(= LAI)$ [resp. right approximate identity $(= RAI)$] if, for each $a \in A$, $e_\alpha a \longrightarrow a$ [resp. $ae_\alpha \longrightarrow a$]. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $< a'a, b >=< a', ab >= a'(ab)$ and $< aa', b >=< a', ba >= a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $< a, a' >$ - $< a', a >$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : \ a \in A \ and \ a' \in A^*\}$ and $\{aa' : \ a \in A \ and \ a' \in A^*\}$ by A^*A and AA^*, respectively, clearly these two sets are subsets of A^*. Let A has a BAI. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides. Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following:

1. $m^* : Z^* \times X \to Y^*$, given by $< m^*(z', x), y >=< z', m(x, y) >$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $< m^{**}(y'', z'), x >=< y'', m^*(z', x) >$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $< m^{***}(x'', y''), z' > =< x'', m^{**}(y'', z') >$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping

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$y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is \ weak^* - to - weak^* - continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is \ weak^* - to - weak^* - continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j < z', m(x_i, y_j) >= \lim_j \lim_i < z', m(x_i, y_j) >,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [6, 18].
The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstin's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta < a', m(a_\alpha, b_\beta) >=< a''b'', a' >$$

and

$$\lim_\beta \lim_\alpha < a', m(a_\alpha, b_\beta) >=< a''ob'', a' >,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [6, 14, 18].
The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.
This paper is organized as follows.
a) In section two, for a Banach $A - bimodule$, we have

(1) $a'' \in Z_{B^{**}}(A^{**})$ if and only if $\pi_\ell^{****}(b', a'') \in B^*$ for all $b' \in B^*$.
(2) $F \in Z_{B^{**}}((A^*A)^*)$ if and only if $\pi_\ell^{****}(g, F) \in B^*$ for all $g \in B^*$.
(3) $G \in Z_{(A^*A)^*}(B^{**})$ if and only if $\pi_r^{****}(g, G) \in A^*A$ for all $g \in B^*$.
(4) Let B has a BAI $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \xrightarrow{w^*} e''$. Then if $Z^t_{e^{**}}(B^{**}) = B^{**}$ [resp. $Z_{e^{**}}(B^{**}) = B^{**}$] and B^* factors on the left [resp. right], but not on the right [resp. left], then $Z_{B^{**}}(A^{**}) \neq Z^t_{B^{**}}(A^{**})$.
(5) $B^*A \subseteq wap_\ell(B)$ if and only if $AA^{**} \subseteq Z_{B^{**}}(A^{**})$.
(6) Let $b' \in B^*$. Then $b' \in wap_\ell(B)$ if and only if the adjoint of the mapping $\pi_\ell^*(b', \ ) : A \to B^*$ is $weak^* - to - weak$ continuous.

b) In section three, for a Banach $A-bimodule$ B, we define $Left-weak^*-to-weak$ property [$=Rw^*w-$ property] and $Right-weak^*-to-weak$ property [$=Rw^*w-$ property] for Banach algebra A and we show that

(1) If $A^{**}=a_0A^{**}$ [resp. $A^{**}=A^{**}a_0$] for some $a_0\in A$ and a_0 has Rw^*w- property [resp. Lw^*w- property], then $Z_{B^{**}}(A^{**})=A^{**}$.
(2) If $B^{**}=a_0B^{**}$ [resp. $B^{**}=B^{**}a_0$] for some $a_0\in A$ and a_0 has Rw^*w- property [resp. Lw^*w- property] with respect to B, then $Z_{A^{**}}(B^{**})=B^{**}$.
(3) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w- property [resp. Lw^*w- property], then $Z_{B^{**}}(A^{**})=A^{**}$.
(4) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w- property [resp. Lw^*w- property] with respect B, then $Z_{A^{**}}(B^{**})=B^{**}$.
(5) If $a_0\in A$ has Rw^*w- property with respect to B, then $a_0A^{**}\subseteq Z_{B^{**}}(A^{**})$ and $a_0B^*\subseteq wap_\ell(B)$.
(6) Assume that $AB^*\subseteq wap_\ell B$. If B^* strong factors on the left [resp. right], then A has Lw^*w- property [resp. Rw^*w- property] with respect to B.
(7) Assume that $AB^*\subseteq wap_\ell B$. If B^* strong factors on the left [resp. right], then A has Lw^*w- property [resp. Rw^*w- property] with respect to B.

2. The topological centers of module actions

Let B be a Banach $A-bimodule$, and let

$$\pi_\ell:\ A\times B\to B \ and\ \pi_r:\ B\times A\to B.$$

be the left and right module actions of A on B. Then B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***}:\ A^{**}\times B^{**}\to B^{**}\ and\ \pi_r^{***}:\ B^{**}\times A^{**}\to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t}:\ A^{**}\times B^{**}\to B^{**}\ and\ \pi_r^{t***t}:\ B^{**}\times A^{**}\to B^{**}.$$

We may therefore define the topological centers of the right and left module actions of A on B as follows:

$$\begin{aligned}Z_{A^{**}}(B^{**})=Z(\pi_r)=\{b''\in B^{**}:\ &the\ map\ a''\to\pi_r^{***}(b'',a'')\ :\ A^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\\ Z_{B^{**}}(A^{**})=Z(\pi_\ell)=\{a''\in A^{**}:\ &the\ map\ b''\to\pi_\ell^{***}(a'',b'')\ :\ B^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\\ Z^t_{A^{**}}(B^{**})=Z(\pi_\ell^t)=\{b''\in B^{**}:\ &the\ map\ a''\to\pi_\ell^{t***}(b'',a'')\ :\ A^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\\ Z^t_{B^{**}}(A^{**})=Z(\pi_r^t)=\{a''\in A^{**}:\ &the\ map\ b''\to\pi_r^{t***}(a'',b'')\ :\ B^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\end{aligned}$$

We note also that if B is a left(resp. right) Banach $A-module$ and $\pi_\ell:\ A\times B\to B$ (resp. $\pi_r:\ B\times A\to B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A-module$.

We write $ab=\pi_\ell(a,b)$, $ba=\pi_r(b,a)$, $\pi_\ell(a_1a_2,b)=\pi_\ell(a_1,a_2b)$, $\pi_r(b,a_1a_2)=\pi_r(ba_1,a_2)$, $\pi_\ell^*(a_1b',a_2)=\pi_\ell^*(b',a_2a_1)$, $\pi_r^*(b'a,b)=\pi_r^*(b',ab)$, for all $a_1,a_2,a\in A$, $b\in B$ and $b'\in B^*$ when there is no confusion.

Theorem 2-1. We have the following assertions.

(1) Assume that B is a Left Banach $A-module$. Then, $a''\in Z_{B^{**}}(A^{**})$ if and only if $\pi_\ell^{****}(b',a'')\in B^*$ for all $b'\in B^*$.
(2) Assume that B is a right Banach $A-module$. Then, $b''\in Z_{A^{**}}(B^{**})$ if and only if $\pi_r^{****}(b',b'')\in A^*$ for all $b'\in B^*$.

Proof. (1) Let $b''\in B^{**}$. Then, for every $a''\in Z_{B^{**}}(A^{**})$, we have

$$< \pi_\ell^{****}(b',a''),b''>=< b',\pi_\ell^{***}(a'',b'')>=<\pi_\ell^{***}(a'',b''),b'>$$
$$=<\pi_\ell^{t****t}(a'',b''),b'>=<\pi_\ell^{t***}(b'',a''),b'>=< b'',\pi_\ell^{t**}(a'',b')> .$$

It follow that $\pi_\ell^{****}(b',a'')=\pi_\ell^{t**}(a'',b')\in B^*$.
Conversely, let $a''\in A^{**}$ and let $\pi_\ell^{****}(a'',b')\in B^*$ for all $b'\in B^*$. Then for all $b''\in B^{**}$, we have

$$<\pi_\ell^{***}(a'',b''),b'>=< b',\pi_\ell^{***}(a'',b'')>=<\pi_\ell^{****}(b',a''),b''>$$
$$=<\pi_\ell^{t**}(a'',b'),b''>=< b'',\pi_\ell^{t**}(a'',b')>=<\pi_\ell^{t***}(b'',a''),b'>$$
$$=<\pi_\ell^{t****t}(a'',b''),b'> .$$

Consequently $a''\in Z_{B^{**}}(A^{**})$.
(2) Prof is similar to (1).

□

Theorem 2-2. Assume that B is a Banach $A-bimodule$. Then we have the following assertions.

(1) $F\in Z_{B^{**}}((A^*A)^*)$ if and only if $\pi_\ell^{****}(g,F)\in B^*$ for all $g\in B^*$.
(2) $G\in Z_{(A^*A)^*}(B^{**})$ if and only if $\pi_r^{****}(g,G)\in A^*A$ for all $g\in B^*$.

Proof. (1) Let $F\in Z_{B^{**}}((A^*A)^*)$ and $(b''_\alpha)_\alpha\subseteq B^{**}$ such that $b''_\alpha\overset{w^*}{\to}b''$. Then for all $g\in B^*$, we have

$$<\pi_\ell^{****}(g,F),b''_\alpha>=< g,\pi_\ell^{***}(F,b''_\alpha)>=<\pi_\ell^{***}(F,b''_\alpha),g>$$
$$\to<\pi_\ell^{***}(F,b''),g>=<\pi_\ell^{****}(g,F),b''> .$$

Thus, we conclude that $\pi_\ell^{****}(g,F)\in(B^{**},weak^*)^*=B^*$.
Conversely, let $\pi_\ell^{****}(g,F)\in B^*$ for $F\in(A^*A)^*$ and $g\in B^*$. Assume that $b''\in B^{**}$ and $(b''_\alpha)_\alpha\subseteq B^{**}$ such that $b''_\alpha\overset{w^*}{\to}b''$. Then

$$<\pi_\ell^{***}(F,b''_\alpha),g>=< g,\pi_\ell^{***}(F,b''_\alpha)>=<\pi_\ell^{****}(g,F),b''_\alpha>$$
$$=< b''_\alpha,\pi_\ell^{****}(g,F)>\to< b'',\pi_\ell^{****}(g,F)>=<\pi_\ell^{****}(g,F),b''>$$
$$=<\pi_\ell^{***}(F,b''),g> .$$

It follow that $F\in Z_{B^{**}}((A^*A)^*)$.
(2) Proof is similar to (1).

□

In the proceeding theorems, if we take $B = A$, we obtain some parts of Lemma 3.1 from [14].

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By [4, p.146], an element e'' of A^{**} is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.
Let B be a Banach $A-bimodule$ and $a'' \in A^{**}$. We define the locally topological center of the left and right module actions of a'' on B, respectively, as follows

$$Z^t_{a''}(B^{**}) = Z^t_{a''}(\pi^t_\ell) = \{b'' \in B^{**} :\ \pi^{t***t}_\ell(a'', b'') = \pi^{***}_\ell(a'', b'')\},$$
$$Z_{a''}(B^{**}) = Z_{a''}(\pi^t_r) = \{b'' \in B^{**} :\ \pi^{t***t}_r(b'', a'') = \pi^{***}_r(b'', a'')\}.$$

Thus we have
$$\bigcap_{a''\in A^{**}} Z^t_{a''}(B^{**}) = Z^t_A(B^{**}) = Z(\pi^t_r),$$
$$\bigcap_{a''\in A^{**}} Z_{a''}(B^{**}) = Z_A(B^{**}) = Z(\pi_r).$$

Definition 2-3. Let B be a left Banach $A-module$ and $e'' \in A^{**}$ be a mixed unit for A^{**}. We say that e'' is a left mixed unit for B^{**}, if
$$\pi^{***}_\ell(e'', b'') = \pi^{t***t}_\ell(e'', b'') = b'',$$
for all $b'' \in B^{**}$.
The definition of right mixed unit for B^{**} is similar. B^{**} has a mixed unit if it has left and right mixed unit that are equal.
It is clear that if $e'' \in A^{**}$ is a left (resp. right) unit for B^{**} and $Z_{e''}(B^{**}) = B^{**}$, then e'' is left (resp. right) mixed unit for B^{**}.

Theorem 2-4. Let B be a Banach $A-bimodule$ with a BAI $(e_\alpha)_\alpha$ such that $e_\alpha \overset{w^*}{\to} e''$. Then if $Z^t_{e^{**}}(B^{**}) = B^{**}$ [resp. $Z_{e^{**}}(B^{**}) = B^{**}$] and B^* factors on the left [resp. right], but not on the right [resp. left], then $Z_{B^{**}}(A^{**}) \neq Z^t_{B^{**}}(A^{**})$.

Proof. Suppose that B^* factors on the left with respect to A, but not on the right. Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \overset{w^*}{\to} e''$. Thus for all $b' \in B^*$ there are $a \in A$ and $x' \in B^*$ such that $x'a = b'$. Then for all $b'' \in B^{**}$ we have

$$< \pi^{***}_\ell(e'', b''), b' >=< e'', \pi^{**}_\ell(b'', b') >= \lim_\alpha < \pi^{**}_\ell(b'', b'), e_\alpha >$$
$$= \lim_\alpha < b'', \pi^*_\ell(b', e_\alpha) >= \lim_\alpha < b'', \pi^*_\ell(x'a, e_\alpha) >$$
$$= \lim_\alpha < b'', \pi^*_\ell(x', ae_\alpha) >= \lim_\alpha < \pi^{**}_\ell(b'', x'), ae_\alpha >$$
$$=< \pi^{**}_\ell(b'', x'), a >=< b'', b' > .$$

Thus $\pi^{***}_\ell(e'', b'') = b''$ consequently B^{**} has left unit $A^{**} - module$. It follows that $e'' \in Z_{B^{**}}(A^{**})$. If we take $Z_{B^{**}}(A^{**}) = Z^t_{B^{**}}(A^{**})$, then $e'' \in Z^t_{B^{**}}(A^{**})$. Then the

mapping $b'' \to \pi_r^{t***t}(b'', e'')$ is $weak^* - to - weak^*$ continuous from B^{**} into B^{**}. Since $e_\alpha \xrightarrow{w^*} e''$, $\pi_r^{t***t}(b'', e_\alpha) \xrightarrow{w^*} \pi_r^{t***t}(b'', e'')$. Let $b' \in B^*$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Since $Z_{e^{**}}^t(B^{**}) = B^{**}$, we have the following quality

$$< \pi_r^{t***t}(b'', e''), b' >= \lim_\alpha < \pi_r^{t***t}(b'', e_\alpha), b' >= \lim_\alpha < \pi_r^{t***}(e_\alpha, b''), b' >$$

$$= \lim_\alpha \lim_\beta < \pi_r^{t***}(e_\alpha, b_\beta), b' >= \lim_\alpha \lim_\beta < \pi_r(b_\beta, e_\alpha), b' >$$

$$= \lim_\alpha \lim_\beta < b', \pi_r(b_\beta, e_\alpha) >= \lim_\beta \lim_\alpha < b', \pi_r(b_\beta, e_\alpha) >$$

$$= \lim_\beta < b', b_\beta >=< b'', b' > .$$

Thus $\pi_r^{t***t}(b'', e'') = \pi_r^{***}(b'', e'') = b''$. It follows that B'' has a right unit. Suppose that $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Then for all $b' \in B^*$ we have

$$< b'', b' >=< \pi_r^{***}(b'', e''), b' >=< b'', \pi_r^{**}(e'', b') >= \lim_\beta < \pi_r^{**}(e'', b'), b_\beta >$$

$$= \lim_\beta < e'', \pi_r^*(b', b_\beta) >= \lim_\beta \lim_\alpha < \pi_r^*(b', b_\beta), e_\alpha >$$

$$= \lim_\beta \lim_\alpha < \pi_r^*(b', b_\beta), e_\alpha >= \lim_\beta \lim_\alpha < b', \pi_r(b_\beta, e_\alpha) >$$

$$= \lim_\alpha \lim_\beta < \pi_r^{***}(b_\beta, e_\alpha), b' >= \lim_\alpha \lim_\beta < b_\beta, \pi_r^{**}(e_\alpha, b') >$$

$$= \lim_\alpha < b'', \pi_r^{**}(e_\alpha, b') > .$$

It follows that $weak - \lim_\alpha \pi_r^{**}(e_\alpha, b') = b'$. So by Cohen Factorization Theorem, B^* factors on the right that is contradiction.

□

Corollary 2-5. Let B be a Banach $A - bimodule$ and $e'' \in A^{**}$ be a left mixed unit for B^{**}. If B^* factors on the left, but not on the right, then $Z_{B^{**}}(A^{**}) \neq Z_{B^{**}}^t(A^{**})$.

In the proceeding corollary, if we take $B = A$, then it is clear $Z_{e^{**}}^t(A^{**}) = A^{**}$, and so we obtain Proposition 2.10 from [14].

Theorem 2-6. Suppose that B is a weakly complete Banach space. Then we have the following assertions.

(1) Let B be a Left Banach $A - module$ and e'' be a left mixed unit for B^{**}. If $AB^{**} \subseteq B$, then B is reflexive.

(2) Let B be a right Banach $A - module$ and e'' be a right mixed unit for B^{**}. If $Z_{A^{**}}(B^{**})A \subseteq B$, then $Z_{A^{**}}(B^{**}) = B$.

Proof. (1) Assume that $b'' \in B^{**}$. Since e'' is also mixed unit for A^{**}, there is a BAI $(e_\alpha)_\alpha \subseteq A$ for A such that $e_\alpha \xrightarrow{w^*} e''$. Then $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} \pi_\ell^{***}(e'', b'') = b''$ in B^{**}. Since $AB^{**} \subseteq B$, we have $\pi_\ell^{***}(e_\alpha, b'') \in B$. Consequently $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w} \pi_\ell^{***}(e'', b'') = b''$ in B. Since B is a weakly complete, $b'' \in B$, and so B is reflexive.

(2) Since $b'' \in Z_{A^{**}}(B^{**})$, we have $\pi_r^{***}(b'', e_\alpha) \xrightarrow{w^*} \pi_r^{***}(b'', e'') = b''$ in B^{**}. Since $Z_{A^{**}}(B^{**})A \subseteq B$, $\pi_r^{***}(b'', e_\alpha) \in B$. Consequently we have $\pi_r^{***}(b'', e_\alpha) \xrightarrow{w} \pi_r^{***}(b'', e'') = b''$ in B. It follows that $b'' \in B$, since B is a weakly complete. $\square$

A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. The procceding definition to the equivalent following condition, see [6, 14, 18].
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \| a \| \leq 1\}$, we have

$$lim_\alpha lim_\beta < a', a_\alpha b_\beta >= lim_\beta lim_\alpha < a', a_\alpha b_\beta >,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $< a''b'', a' >=< a''ob'', a' >$ for every $a'',\ b'' \in A^{**}$.

Definition 2-7. Let B be a left Banach $A-module$. Then, $b' \in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell(b', a) :\ a \in A,\ \| a \| \leq 1\}$ is relatively weakly compact. We denote by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^*.
The definition of the right weakly almost periodic functional ($= wap_r(B)$) is the same. By [18], the definition of $wap_\ell(B)$ is equivalent to the following

$$< \pi_\ell^{***}(a'', b''), b' >=< \pi_\ell^{t***t}(a'', b''), b' >$$

for all $a'' \in A^{**}$ and $b'' \in B^{**}$. Thus, we can write

$$wap_\ell(B) = \{b' \in B^* : < \pi_\ell^{***}(a'', b''), b' >=< \pi_\ell^{t***t}(a'', b''), b' > \\ for\ all\ a'' \in A^{**},\ b'' \in B^{**}\}.$$

Theorem 2-8. Suppose that B is a left Banach $A-module$. Consider the following statements.

(1) $B^*A \subseteq wap_\ell(B)$.
(2) $AA^{**} \subseteq Z_{B^{**}}(A^{**})$.
(3) $AA^{**} \subseteq AZ_{B^{**}}((A^*A)^*)$.

Then, we have (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3).

Proof. (1) $\Rightarrow$ (2)
Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $a \in A$ and $a'' \in A^{**}$, we have

$$< \pi_\ell^{***}(aa'', b''_\alpha), b' >=< aa'', \pi_\ell^{**}(b''_\alpha, b') >=< a'', \pi_\ell^{**}(b''_\alpha, b')a >$$
$$=< a'', \pi_\ell^{**}(b''_\alpha, b'a) >=< \pi_\ell^{***}(a'', b''_\alpha), b'a) >\to< \pi_\ell^{***}(a'', b''), b'a) >$$
$$=< \pi_\ell^{***}(aa'', b''), b') > .$$

Hence $aa'' \in Z_{B^{**}}(A^{**})$.
(2) $\Rightarrow$ (1)
Let $a \in A$ and $b' \in B^*$. Then

$$< \pi_\ell^{***}(a'', b''_\alpha), b'a >=< a\pi_\ell^{***}(a'', b''_\alpha), b' >=< \pi_\ell^{***}(aa'', b''_\alpha), b' >$$

$$=<\pi_\ell^{t***t}(aa'',b''_\alpha),b'>=<\pi_\ell^{t***t}(a'',b''_\alpha),b'a>.$$

It follow that $b'a\in wap_\ell(B)$.
(3) $\Rightarrow$ (2)
Since $AZ_{B^{**}}((A^*A)^*)\subseteq Z_{B^{**}}(A^{**})$, proof is hold. $\square$

In the proceeding theorem, if we take $B=A$, then we obtain Theorem 3.6 from [14] and the same as proceeding theorem, we can claim the following assertions:
If B is a right Banach $A-module$, then for the following statements we have (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3).

(1) $AB^*\subseteq wap_r(B)$.
(2) $A^{**}A\subseteq Z_{B^{**}}(A^{**})$.
(3) $A^{**}A\subseteq Z_{B^{**}}((A^*A)^*)A$.

The proof of the this assertion is similar to proof of Theorem 2-8.

Corollary 2-9. Suppose that B is a Banach $A-bimodule$. Then if A is a left [resp. right] ideal in A^{**}, then $B^*A\subseteq wap_\ell(B)$ [resp. $AB^*\subseteq wap_r(B)$].

Example 2-10. Suppose that $1\leq p\leq\infty$ and q is conjugate of p. We know that if G is compact, then $L^1(G)$ is a two-sided ideal in its second dual of it. By proceeding Theorem we have $L^q(G)*L^1(G)\subseteq wap_\ell(L^p(G))$ and $L^1(G)*L^q(G)\subseteq wap_r(L^p(G))$. Also if G is finite, then $L^q(G)\subseteq wap_\ell(L^p(G))\cap wap_r(L^p(G))$. Hence we conclude that

$$Z_{L^1(G)^{**}}(L^p(G)^{**})=L^p(G)\ and\ Z_{L^p(G)^{**}}(L^1(G)^{**})=L^1(G).$$

Theorem 2-11. We have the following assertions.

(1) Suppose that B is a left Banach $A-module$ and $b'\in B^*$. Then $b'\in wap_\ell(B)$ if and only if the adjoint of the mapping $\pi_\ell^*(b',\ ):A\rightarrow B^*$ is $weak^*-to-weak$ continuous.
(2) Suppose that B is a right Banach $A-module$ and $b'\in B^*$. Then $b'\in wap_r(B)$ if and only if the adjoint of the mapping $\pi_r^*(b',\ ):B\rightarrow A^*$ is $weak^*-to-weak$ continuous.

Proof. (1) Assume that $b'\in wap_\ell(B)$ and $\pi_\ell^*(b',\ )^*:B^{**}\rightarrow A^*$ is the adjoint of $\pi_\ell^*(b',\ )$. Then for every $b''\in B^{**}$ and $a\in A$, we have

$$<\pi_\ell^*(b',\ )^*b'',a>=<b'',\pi_\ell^*(b',a)>.$$

Suppose $(b''_\alpha)_\alpha\subseteq B^{**}$ such that $b''_\alpha\overset{w^*}{\rightarrow}b''$ and $a''\in A^{**}$ and $(a_\beta)_\beta\subseteq A$ such that $a_\beta\overset{w^*}{\rightarrow}a''$. By easy calculation, for all $y''\in B^{**}$ and $y'\in B^*$, we have

$$<\pi_\ell^*(y',\ )^*,y''>=\pi_\ell^{**}(y'',y').$$

Since $b'\in wap_\ell(B)$,

$$<\pi_\ell^{***}(a'',b''_\alpha),b'>\rightarrow<\pi_\ell^{***}(a'',b''),b'>.$$

Then we have the following statements

$$\lim_\alpha<a'',\pi_\ell^*(b',\ )^*b''_\alpha>=\lim_\alpha<a'',\pi_\ell^{**}(b''_\alpha,b')>

$$= \lim_{\alpha} < \pi_\ell^{***}(a'', b''_\alpha), b' >=< \pi_\ell^{***}(a'', b''), b' >$$

$$=< a'', \pi_\ell^{*}(b', \ )^* b'' > .$$

It follow that the adjoint of the mapping $\pi_\ell^*(b', \ ) : A \rightarrow B^*$ is $weak^* - to - weak$ continuous.

Conversely, let the adjoint of the mapping $\pi_\ell^*(b', \ ) : A \rightarrow B^*$ is $weak^* - to - weak$ continuous. Suppose $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \stackrel{w^*}{\rightarrow} b''$ and $b' \in B^*$. Then for every $a'' \in A^{**}$, we have

$$\lim_{\alpha} < \pi_\ell^{***}(a'', b''_\alpha), b' >= \lim_{\alpha} < a'', \pi_\ell^{**}(b''_\alpha, b') >$$

$$= \lim_{\alpha} < a'', \pi_\ell^{*}(b', \ )^* b''_\alpha >=< a'', \pi_\ell^{*}(b', \ )^* b'' >=< \pi_\ell^{***}(a'', b''), b' > .$$

It follow that $b' \in wap_\ell(B)$.

(2) proof is similar to (1).

□

Corollary 2-12. Let A be a Banach algebra. Assume that $a' \in A^*$ and $T_{a'}$ is the linear operator from A into A^* defined by $T_{a'}a = a'a$. Then, $a' \in wap(A)$ if and only if the adjoint of $T_{a'}$ is $weak^* - to - weak$ continuous. So A is Arens regular if and only if the adjoint of the mapping $T_{a'}a = a'a$ is $weak^* - to - weak$ continuous for every $a' \in A^*$.

3. Lw^*w-property and Rw^*w-property

In this section, we introduce the new definition as $Left - weak^* - to - weak$ property and $Right - weak^* - to - weak$ property for Banach algebra A and make some relations between these concepts and topological centers of module actions. As some conclusion, we have $Z_{L^1(G)^{**}}(M(G)^{**}) \neq M(G)^{**}$ where G is a locally compact group. If G is finite, we have $Z_{M(G)^{**}}(L^1(G)^{**}) = L^1(G)^{**}$ and $Z_{L^1(G)^{**}}(M(G)^{**}) = M(G)^{**}$.

Definition 3-1. Let B be a left Banach $A - module$. We say that $a \in A$ has $Left - weak^* - to - weak$ property ($= Lw^*w-$ property) with respect to B, if for all $(b_\alpha)_\alpha \subseteq B^*$, $ab'_\alpha \stackrel{w^*}{\rightarrow} 0$ implies $ab'_\alpha \stackrel{w}{\rightarrow} 0$. If every $a \in A$ has Lw^*w- property with respect to B, then we say that A has Lw^*w- property with respect to B. The definition of the $Right - weak^* - to - weak$ property ($= Rw^*w-$ property) is the same.

We say that $a \in A$ has $weak^* - to - weak$ property ($= w^*w-$ property) with respect to B if it has Lw^*w- property and Rw^*w- property with respect to B.

If $a \in A$ has Lw^*w- property with respect to itself, then we say that $a \in A$ has Lw^*w- property.

For proceeding definition, we have some examples and remarks as follows.

a) If B is Banach A-bimodule and reflexive, then A has w^*w-property with respect to B. Then

i) $L^1(G)$, $M(G)$ and $A(G)$ have w^*w-property when G is finite.

ii) Let G be locally compact group. $L^1(G)$ [resp. $M(G)$] has w^*w-property [resp.

Lw^*w- property]with respect to $L^p(G)$ whenever $p > 1$.
b) Suppose that B is a left Banach $A-module$ and e is left unit element of A such that $eb = b$ for all $b \in B$. If e has Lw^*w- property, then B is reflexive.
c) If S is a compact semigroup, then $C^+(S) = \{f \in C(S) : \ f > 0\}$ has w^*w-property.

Theorem 3-2. Suppose that B is a Banach $A - bimodule$. Then we have the following assertions.

(1) If $A^{**} = a_0A^{**}$ [resp. $A^{**} = A^{**}a_0$] for some $a_0 \in A$ and a_0 has Rw^*w- property [resp. Lw^*w- property], then $Z_{B^{**}}(A^{**}) = A^{**}$.
(2) If $B^{**} = a_0B^{**}$ [resp. $B^{**} = B^{**}a_0$] for some $a_0 \in A$ and a_0 has Rw^*w- property [resp. Lw^*w- property] with respect to B, then $Z_{A^{**}}(B^{**}) = B^{**}$.

Proof. (1) Suppose that $A^{**} = a_0A^{**}$ for some $a_0 \in A$ and a_0 has Rw^*w- property. Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \stackrel{w^*}{\to} b''$. Then for all $a \in A$ and $b' \in B^*$, we have

$$< \pi_\ell^{**}(b''_\alpha, b'), a >=< b''_\alpha, \pi_\ell^*(b', a) >\to< b'', \pi_\ell^*(b', a) >=< \pi_\ell^{**}(b'', b'), a >,$$

it follow that $\pi_\ell^{**}(b''_\alpha, b') \stackrel{w^*}{\to} \pi_\ell^{**}(b'', b')$. Also we can write $\pi_\ell^{**}(b''_\alpha, b')a_0 \stackrel{w^*}{\to} \pi_\ell^{**}(b'', b')a_0$. Since a_0 has Rw^*w- property, $\pi_\ell^{**}(b''_\alpha, b')a_0 \stackrel{w}{\to} \pi_\ell^{**}(b'', b')a_0$. Now let $a'' \in A^{**}$. Then there is $x'' \in A^{**}$ such that $a'' = a_0x''$ consequently we have

$$< \pi_\ell^{***}(a'', b''_\alpha), b' >=< a'', \pi_\ell^{**}(b''_\alpha, b') >=< x'', \pi_\ell^{**}(b''_\alpha, b')a_0 >$$
$$\to< x'', \pi_\ell^{**}(b'', b')a_0 >=< \pi_\ell^{***}(a'', b''_\alpha), b' > .$$

We conclude that $a'' \in Z_{B^{**}}(A^{**})$. Proof of the next part is the same as the proceeding proof.

(2) Let $B^{**} = a_0B^{**}$ for some $a_0 \in A$ and a_0 has Rw^*w- property with respect to B. Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \stackrel{w^*}{\to} a''$. Then for all $b \in B$, we have

$$< \pi_r^{**}(a''_\alpha, b'), b >=< a''_\alpha, \pi_r^{**}(b', b) >\to< a'', \pi_r^{**}(b', b) >=< \pi_r^{**}(a'', b'), b > .$$

We conclude that $\pi_r^{**}(a''_\alpha, b') \stackrel{w^*}{\to} \pi_r^{**}(a'', b')$ then we have $\pi_r^{**}(a''_\alpha, b')a_0 \stackrel{w^*}{\to} \pi_r^{**}(a'', b')a_0$. Since a_0 has Rw^*w- property with respect to B, $\pi_r^{**}(a''_\alpha, b')a_0 \stackrel{w}{\to} \pi_r^{**}(a'', b')a_0$.
Now let $b'' \in B^{**}$. Then there is $x'' \in B^{**}$ such that $b'' = a_0x''$. Hence, we have

$$< \pi_r^{***}(b'', a''_\alpha), b' >=< b'', \pi_r^{**}(a''_\alpha, b') >=< a_0x'', \pi_r^{**}(a''_\alpha, b') >$$
$$=< x'', \pi_r^{**}(a''_\alpha, b')a_0 >\to< x'', \pi_r^{**}(a'', b')a_0 >=< b'', \pi_r^{**}(a'', b') >$$
$$=< \pi_r^{***}(b'', a''), b' > .$$

It follow that $b'' \in Z_{A^{**}}(B^{**})$. The next part is similar to the proceeding proof.

□

Example 3-3. i) Let G be a locally compact group. Since $M(G)$ is a Banach $L^1(G)$-bimodule and the unit element of $M(G)$ has not Lw^*w- property or Rw^*w- property, by Theorem 2-3, $Z_{L^1(G)^{**}}(M(G)^{**}) \neq M(G)^{**}$.
ii) If G is finite, then by Theorem 2-3, we have $Z_{M(G)^{**}}(L^1(G)^{**}) = L^1(G)^{**}$ and $Z_{L^1(G)^{**}}(M(G)^{**}) = M(G)^{**}$.

Assume that B is a Banach $A-bimodule$. We say that B factors on the left (right) with respect to A if $B = BA$ ($B = AB$). We say that B factors on both sides, if $B = BA = AB$.

Theorem 3-4. Suppose that B is a Banach $A-bimodule$ and A has a BAI. Then we have the following assertions.

(1) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w- property [resp. Lw^*w- property], then $Z_{B^{**}}(A^{**}) = A^{**}$.
(2) If B^* factors on the left [resp. right] with respect to A and A has Rw^*w- property [resp. Lw^*w- property] with respect B, then $Z_{A^{**}}(B^{**}) = B^{**}$.

Proof. (1) Assume that B^* factors on the left and A has Rw^*w- property. Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \overset{w^*}{\to} b''$. Since $B^*A = B^*$, for all $b' \in B^*$ there are $x \in A$ and $y' \in B^*$ such that $b' = y'x$. Then for all $a \in A$, we have

$$< \pi_\ell^{**}(b''_\alpha, y')x, a >=< b''_\alpha, \pi_\ell^*(y', a)x >=< \pi_\ell^{**}(b''_\alpha, b'), a >$$
$$=< b''_\alpha, \pi_\ell^*(b', a) >\to< b'', \pi_\ell^*(b', a) >=< \pi_\ell^{**}(b'', y')x, a > .$$

Thus, we conclude that $\pi_\ell^{**}(b''_\alpha, y')x \overset{w^*}{\to}< \pi_\ell^{**}(b'', y')x$. Since A has Rw^*w- property, $\pi_\ell^{**}(b''_\alpha, y')x \overset{w}{\to}< \pi_\ell^{**}(b'', y')x$. Now let $b'' \in A^{**}$. Then

$$< \pi_\ell^{***}(a'', b''_\alpha), b' >=< a'', \pi_\ell^{**}(b''_\alpha, b') >=< a'', \pi_\ell^{**}(b''_\alpha, y')x >$$
$$\to< a'', \pi_\ell^{**}(b'', y')x >=< \pi_\ell^{***}(a'', b''), b' > .$$

It follow that $a'' \in Z_{B^{**}}(A^{**}) = A^{**}$.
If B^* factors on the right and A has Lw^*w- property, then proof is the same as preceding proof.

(2) Let B^* factors on the left with respect to A and A has Rw^*w- property with respect to B. Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \overset{w^*}{\to} a''$. Since $B^*A = B$, for all $b' \in B^*$ there are $x \in A$ and $y' \in B^*$ such that $b' = y'x$. Then for all $b \in B$, we have

$$< \pi_r^{**}(a''_\alpha, y')x, b >=< \pi_r^{**}(a''_\alpha, b'), b >=< a''_\alpha, \pi_r^*(b', b) >$$
$$=< a'', \pi_r^*(b', b) >=< \pi_r^{**}(a'', y')x, b > .$$

Consequently $\pi_r^{**}(a''_\alpha, y')x \overset{w^*}{\to} \pi_r^{**}(a'', y')x$. Since A has Rw^*w- property with respect to B, $\pi_r^{**}(a''_\alpha, y')x \overset{w}{\to} \pi_r^{**}(a'', y')x$. It follow that for all $b'' \in B^{**}$, we have

$$< \pi_r^{***}(b'', a''_\alpha), b' >=< b'', \pi_r^{**}(a''_\alpha, y')x >\to< b'', \pi_r^{**}(a'', y')x >$$
$$=< \pi_r^{***}(b'', a''), b' > .$$

Thus we conclude that $b'' \in Z_{A^{**}}(B^{**})$.
The proof of the next assertions is the same as proceeding proof.

$\square$

Theorem 3-5. Suppose that B is a Banach $A-bimodule$. Then we have the following assertions.

(1) If $a_0 \in A$ has Rw^*w- property with respect to B, then $a_0A^{**} \subseteq Z_{B^{**}}(A^{**})$ and $a_0B^* \subseteq wap_\ell(B)$.
(2) If $a_0 \in A$ has Lw^*w- property with respect to B, then $A^{**}a_0 \subseteq Z_{B^{**}}(A^{**})$ and $B^*a_0 \subseteq wap_\ell(B)$.
(3) If $a_0 \in A$ has Rw^*w- property with respect to B, then $a_0B^{**} \subseteq Z_{A^{**}}(B^{**})$ and $B^*a_0 \subseteq wap_r(B)$.
(4) If $a_0 \in A$ has Lw^*w- property with respect to B, then $B^{**}a_0 \subseteq Z_{A^{**}}(B^{**})$ and $a_0B^* \subseteq wap_r(B)$.

Proof. (1) Let $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''$. Then for all $a \in A$ and $b' \in B^*$, we have

$$< \pi_\ell^{**}(b''_\alpha, b')a_0, a >=< \pi_\ell^{**}(b''_\alpha, b'), a_0a >=< b''_\alpha, \pi_\ell^*(b', a_0a) >$$

$$\rightarrow < b'', \pi_\ell^*(b', a_0a) >=< \pi_\ell^{**}(b'', b')a_0, a > .$$

It follow that $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w^*} \pi_\ell^{**}(b'', b')a_0$. Since a_0 has Rw^*w- property with respect to B, $\pi_\ell^{**}(b''_\alpha, b')a_0 \xrightarrow{w} \pi_\ell^{**}(b'', b')a_0$.
We conclude that $a_0a'' \in Z_{B^{**}}(A^{**})$ so that $a_0A^{**} \in Z_{B^{**}}(A^{**})$. Since $\pi_\ell^{**}(b'', b')a_0 = \pi_\ell^{**}(b'', b'a_0)$, $a_0B^* \subseteq wap_\ell(B)$.

(2) proof is similar to (1).

(3) Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. Let $b \in B$ and $b' \in B^*$. Then we have

$$< \pi_r^{**}(a''_\alpha, b')a_0, b >=< \pi_r^{**}(a''_\alpha, b'), a_0b >=< a''_\alpha, \pi_r^*(b', a_0b) >$$

$$\rightarrow < a'', \pi_r^*(b', a_0b) >=< \pi_r^{**}(a'', b')a_0, b > .$$

Thus we conclude $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w^*} \pi_r^{**}(a'', b')a_0$. Since a_0 has Rw^*w- property with respect to B, $\pi_r^{**}(a''_\alpha, b')a_0 \xrightarrow{w} \pi_r^{**}(a'', b')a_0$. If $b'' \in B^{**}$, then we have

$$< \pi_r^{***}(a_0b'', a''_\alpha), b' >=< a_0b'', \pi_r^{***}(a''_\alpha, b') >=< b'', \pi_r^{**}(a''_\alpha, b')a_0 >$$

$$=< b'', \pi_r^{**}(a''_\alpha, b')a_0 >=< \pi_r^{***}(a_0b'', a''), b' > .$$

It follow that $a_0b'' \in Z_{A^{**}}(B^{**})$. Consequently we have $a_0B^{**} \in Z_{A^{**}}(B^{**})$. The proof of the next assertion is clear.

(4) Proof is similar to (3).

$\square$

Theorem 3-6. Let B be a Banach $A-bimodule$. Then we have the following assertions.

(1) Suppose
$$\lim_\alpha \lim_\beta < b'_\beta, b_\alpha >= \lim_\beta \lim_\alpha < b'_\beta, b_\alpha >,$$
for every $(b_\alpha)_\alpha \subseteq B$ and $(b'_\beta)_\beta \subseteq B^*$. Then A has Lw^*w- property and Rw^*w- property with respect to B.

(2) If for some $a \in A$,
$$\lim_\alpha \lim_\beta < ab'_\beta, b_\alpha >= \lim_\beta \lim_\alpha < ab'_\beta, b_\alpha >,$$
for every $(b_\alpha)_\alpha \subseteq B$ and $(b'_\beta)_\beta \subseteq B^*$, then a has Rw^*w- property with respect to B. Also if for some $a \in A$,
$$\lim_\alpha \lim_\beta < b'_\beta a, b_\alpha >= \lim_\beta \lim_\alpha < b'_\beta a, b_\alpha >,$$
for every $(b_\alpha)_\alpha \subseteq B$ and $(b'_\beta)_\beta \subseteq B^*$, then a has Lw^*w- property with respect to B.

Proof. (1) Assume that $a \in A$ such that $ab'_\beta \overset{w^*}{\to} 0$ where $(b'_\beta)_\beta \subseteq B^*$. Let $b'' \in B^{**}$ and $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \overset{w^*}{\to} b''$. Then
$$\lim_\beta < b'', ab'_\beta >= \lim_\beta \lim_\alpha < b_\alpha, ab'_\beta >= \lim_\beta \lim_\alpha < ab'_\beta, b_\alpha >$$
$$= \lim_\alpha \lim_\beta < ab'_\beta, b_\alpha >= 0.$$
We conclude that $ab'_\beta \overset{w}{\to} 0$, so A has Lw^*w- property. It also easy that A has Rw^*w- property.

(2) Proof is easy and is the same as (1).

□

Definition 3-7. Let B be a left Banach $A-module$. We say that B^* strong factors on the left [resp. right] if for all $(b'_\alpha)_\alpha \subseteq B^*$ there are $(a_\alpha)_\alpha \subseteq A$ and $b' \in B^*$ such that $b'_\alpha = b'a_\alpha$ [resp. $b'_\alpha = a_\alpha b'$] where $(a_\alpha)_\alpha$ has limit the $weak^*$ topology in A^{**}.
If B^* strong factors on the left and right, then we say that B^* strong factors on the both side.
It is clear that if B^* strong factors on the left [resp. right], then B^* factors on the left [resp. right].

Theorem 3-8. Suppose that B is a Banach $A-bimodule$. Assume that $AB^* \subseteq wap_\ell B$. If B^* strong factors on the left [resp. right], then A has Lw^*w- property [resp. Rw^*w- property] with respect to B.

Proof. Let $(b'_\alpha)_\alpha \subseteq B^*$ such that $ab'_\alpha \overset{w^*}{\to} 0$. Since B^* strong factors on the left, there are $(a_\alpha)_\alpha \subseteq A$ and $b' \in B^*$ such that $b'_\alpha = b'a_\alpha$. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such

that $b_\beta \xrightarrow{w^*} b''$. Then we have

$$\lim_\alpha < b'', ab'_\alpha >= \lim_\alpha \lim_\beta < b_\beta, ab'_\alpha >= \lim_\alpha \lim_\beta < ab'_\alpha, b_\beta >$$

$$= \lim_\alpha \lim_\beta < ab' a_\alpha, b_\beta >= \lim_\alpha \lim_\beta < ab', a_\alpha b_\beta >$$

$$= \lim_\beta \lim_\alpha < ab', a_\alpha b_\beta >= \lim_\beta \lim_\alpha < ab'_\alpha, b_\beta >= 0$$

It follow that $ab'_\alpha \xrightarrow{w} 0$. $\square$

Problems .

(1) Suppose that B is a Banach $A-bimodule$. If B is left or right factors with respect to A, dose A has Lw^*w-property or Rw^*w-property, respectively?
(2) Suppose that B is a Banach $A-bimodule$. Let A has Lw^*w-property with respect to B. Dose $Z_{B^{**}}(A^{**}) = A^{**}$?

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